So I wrote this in LaTeX cause it's the easiest way to get all of the mathematical functions on the computer. This is just a guide ofthe concepts we covered in BC3. It might be useful to reference if you're stuck with a problem.

Some definitions: A series is an infinite polynomial: $\sum_{k=1}^{\infty} k^2$
A sequence is an ordered list of numbers.
Both sequences and series can converge (they are exactly or approximately some number) or diverge. A series can be expressed a a sequence of partial sums. As such, the value of series is the limit of its partial sums.

To determine if a sequence converges or diverges, we learned the following techniques:
Take the limit. If we have a sequence a_k where $a_k = \frac{1}{k}$, we can do the following:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$$

which shows that this sequence converges to 0.

Another possible method for determining if a sequence converges or not is bounded monotone convergence. The theorem for bounded monotone convergence states that with a sequence that is bounded ($m \leq a_k \leq M$ for all k) and monotone (always either increasing or decreasing) then the sequence converges.
This means that to show that a sequence is converging, show that it is bounded (by indicating that it has some maximum and minimum) and monotone. There are a variety of ways to show the sequence is monotone; you can show that the difference between consectuive terms is always positive or negative or that the ratio of consecutive terms is always greater than 1 or less than 1.
Note that you must show that the $n+1$ term follows the necessary pattern, not just any two or three terms.

There are several types of series. We know how to work with a few of them. First of all, geometric series. These series have some observable ratio and take the form

$$\sum_{k=0}^{\infty} a \cdot r^k$$

where a is some constant, r is the ratio, and k is the variable of the summation. The value of the ratio determines the outcome of the value of the series.

If $|r| < 1$, then the series $S = \dfrac{a}{1-r}$
If $r > 1$, then the series $S = +\infty$ or $-\infty$
If $r < 1$, then the series does not exist
If $r = 1$, then the series $S = 0$, $+\infty$, or $-\infty$

The harmonic series $(\sum_{k=1}^{\infty} \dfrac{1}{k})$, which isn't exactly a geometric series, diverges. From this we get the idea of the nth term test, whish states that if $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges. Not that this doesn't say what happens when the limit does equal 0; it only says what happens if the limit does not equal 0.

There are some test that work for series and integrals (namely p-test, comparison, and absolute convergence)

The p-test for improper integrals is easily understood and used. The function must take the form $\frac{a}{x^p}$, where a can be any constant (though it's usually 1), x is the variable of summation, and p is some number. The p-test exists in 3 conditions:

When the limits are from 1 to ∞:
If $p < 1$, then the integral goes to ∞ and the series diverges
If $p > 1$, then the integral is equal to $\frac{-1}{-p+1}$ and the series converges
If $p = 1$, then the integral goes to ∞ and the series diverges

When the limits are from 0 to 1 (this is only for the integral, not the series):
If $p > 1$, then the integral goes to ∞
If $p < 1$, then the integral is equal to $\frac{1}{-p+1}$
If $p = 1$, then the integral goes to ∞

The comparison for test integrals states that with positive functions $f(x)$ and $g(x)$, there exists a relationship such that $f(x) \leq g(x)$. In this case, if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges, and if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges. This is easily understood by considering the image of the graphs of two functions $f(x)$ and $g(x)$ where such a relationship exists. The integral of the smaller function, if it diverges, has infinite area, and so the integral of the larger function must contain that infinite area and some more. On the other hand, if the integral of the larger function converges, then it contains a finite amount of area, so the integral of the smaller function cannot have an inifinite amount of area. This works in a similar manner for series (as we'll see later on)

Absolute convergence is the last test we have for integrals. It states that if $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges as well. Once again, consider the what the graph would look like. If we have some function that goes above and below the x-axis, and the absolute value of its integral converges, and $\int_a^\infty f(x)\,dx < \int_a^\infty |f(x)|\,dx$, then by comparison test $\int_a^\infty f(x)\,dx$ must also converge. Note that because of the nature of comparison test, the opposite (about divergence) does not hold true.

With series, we have a considerably larger number of tests to work with. They are as follows:

nth term test
Requirements: Take the limit of the function as the variable of summation goes to ∞
Process: As above
Works well for: An initial check. For the possibility of converging, a series must pass nth term test, or it has to diverge.

Integral test
Requirements: The terms are positive, continuous, and decreasing (can be shown by taking a derivative, finding the ratio of consectuive terms to be less than 1, etc.
Process: Take the integral of a_k. If it converges/diverges, then $\sum a_k$ does the same thing.
Works well for: Any postive, continuous, decreasing function that you can take the integral of. Also allows you to find the bounds of a series; the lower bound is the integral of a_k and the upper bound is the intergral of a_k plus the first term of the series, a_1.

Comparison test

Requirements: a_k is positive.
Process: To see if the series diverges, take $\sum x_k$ such that $x_k < a_k$. To see if the series converges, take $\sum x_k$ such that $x_k > a_k$
Works well for: Series that we can easily compare to series that we already know converge/diverge, such as $\frac{1}{k}$.

Ratio test

Requirements: None really, but having expressions such as x^k, $x!$, and k^x makes the test much more useful.
Process: Take a limit of the absolute value of the term plus 1 over the original term:

$$L = \lim_{k \to \infty} |\frac{k+1}{k}|$$

If $L > 1$, then the series diverges. If $L = 1$, then the test is inconclusive. If $L < 1$, then the series converges.
Works well for: Series with exponents and factorials, as they are easy to cancel out when divided by one another.

Root test

Requirements: Same as Ratio test.
Process: Take a limit such that:

$$L = \lim_{k \to \infty} \sqrt{|a_k|}$$

The results are read the same way as the ratio test. Note that you don't take the square root, you take the kth root.
Works well for: Series with factorials and powers cancel easily. Note that $n!$ can be approximated to $(\frac{n}{e})^n$

Alternating series test

Requirements: a_k includes the term $(-1)^k$ somewhere. The series must also be decreasing, and $\lim\limits_{k \to \infty} a_k = 0$.
Process: If the requirements are true, then the series converges.
Works well for: Any series with the term $(-1)^k$ in it.

Absolute convergence

This isn't so much of a test as a tool. If AST (above) works, then the series converges conditionally. To find out if it converges absolutely, taking $\sum |a_k|$ and checking for convergence. Absolute convergence also works in places where you may be unable to fulfill the requirements of AST, but still have the term $(-1)^k$.

Finding the exact value of a series can be challenging. Often we calculate with some error. With the integral test, we can calculate the approximate value of a series, because the series follows the integral. Alternating series test also offers a method by the alternating series remainder estimate.
Alternating series remainder estimate works by expressing the series as a sequence of partial sums. Note that AST must work first in order for you to use the alternating series remainder estimate. We take the partial sum S:

$$S_n \leq S \leq S_{n+1}$$

which we can rewrite to:

$$0 \leq S - S_n \leq S_{n+1} - S_n$$

where the last part $(S_{n+1} - S_n)$ is equal to a_{n+1}. Then, we know that:

$$|S - S_n| \leq a_{n+1}$$

This expression is the alternating series remainder estimate. So when given a series to estimate within some number, do the following (q is the number we're told to approximate within):

$$\sum |a_{k+1}| \leq q$$

Solving for the variable gives you the number of terms you need to approximate the series within that error.

Before we go to Taylor Series, note that we can work with some types of telescoping series by expanding them and canceling terms to find the resulting finite number of terms which we can sum to get an exact number as the value of the series.

A Taylor series is a power series representation of a function. It serves to approximate any infinitely differentiable function. Some Taylor Series are adapted from the basic form

$$\sqrt{1+x} = \sum_{k=0}^{\infty} x^k$$

To create the Taylor polynomial, we merely write out each of the terms of the series (because series are infintie polynomials). Note that there are two methods of asking for the Taylor polynomial. One method is to ask for a certain number of terms, in which case, you list the number of terms requested. The other method is to request the nth degree Taylor polynomial, in which case the number of terms you write out equals $n + 1$.

Before going further with Taylor series and polynomials, lets cover some things about power series, because all Taylor series are power series. Power series take the general form

$$\sum_{k=0}^{\infty} a_k \cdot x^k$$

Power series also have a radius of convergence, outside of which they will not converge. Taking the derivative and antiderivative of a power series results in the same radius of convergence. Note that when a power series represents a function, the derivative of the power series and function are still equal to one another. The radius of convergence is subject to change when the fuction or power series is changed in other ways.
We know of two ways to find the interval of convergence. If the power series can be expressed as a geometric series, then we can find the ratio and see when the absolute value of the ratio is less than 1. Otherwise, the interval of convergence can be found through ratio/root test. Both are viable methods, and both will tell you the interval of convergence.
We can transform a function in multiple ways. Substitution, multiplication, and differentiating or antidifferentiating all maintain the equality between a power series and a function. With this all said, lets go back to Taylor series.

Taylor's Formula is as follows: Let $f(x)$ be a function which is infinitely differentiable. Let

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(c)}{k!} \cdot (x-c)^k$$

Then

$$|f(x) - P_n(x)| \leq |\frac{K_{n+1}}{(n+1)!} \cdot (x-c)^{n+1}|$$

where

$$|K_{n+1}| \geq |f^n(z)|$$

for z between x and c.

Some of you may have learned these variables differenly. In this case, c is the center (for a MacLaurin polynomial, $c = 0$). k refers to the number of terms, not to be confused with K, which is used in calculating the error. The first ineqaulity above is the method for calculating error with any given Taylor Polynomial (hence the difference between the function and the polynomial approximation).

Finding any given Taylor polynomial is easy enough. First, take the derivative of your function several times until you get a general idea of what you're working with. Then, evaluate those derivatives at your given c value. You should see some sort of pattern. With this pattern, you can substitute these values into your Taylor polynomial and write out the terms. Don't forget to include the x term and divide by a factorial.

There are some Taylor series that you should know, just off the top of your head:

$$\sqrt{1+x} = \sum_{k=0}^{\infty} x^k$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (x)^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (x)^{2k}}{(2k)!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Note that because these are all power series, we can transform them easily enough to find new power series for a different, yet related, function. Also note a new way of finding limits - with power series. If asked for $\lim_{x \to 0} \frac{x \cdot \sin(x)}{\cos(x)}$, you can now express both sine and cosine as series. To solve this limit, merely substitute the series for the respective functions in, and see where you can cancel terms, until you get a finite number of terms that you can work with.

We covered vectors briefly. Vectors can be worked with in the same way as parametric functions; it's just different notation. With parametrics, we were able to differentiate to find speed with the following equation:

$$S = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

Note that differentiating this once more results in velocity, and integrating this results in distance.

Another topic we covered was polar. Know how to graph a polar function - this is easily done by evaluating the given polar function at $\theta = 0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, etc.

Differentiating in polar follows the formula:

$$\frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\cdot\sin(\theta) + f(\theta)\cdot\cos(\theta)}{f'(\theta)\cdot\cos(\theta) + f(\theta)\cdot\sin(\theta)}$$

This is explained by converting everything from polar terms to Cartesian coordinate terms. $x = r\cdot\cos(\theta) = f(\theta)\cdot\cos(\theta)$ and $y = r\cdot\sin(\theta) = f(\theta)\cdot\sin(\theta)$. Differentianting these with respect to θ leads to

$$\frac{dx}{d\theta} = f'(\theta)\cdot\sin(\theta) + f(\theta)\cdot\cos(\theta)$$

and

$$\frac{dy}{d\theta} = f'(\theta)\cdot\cos(\theta) + f(\theta)\cdot\sin(\theta)$$

Dividing these brings us the formula above.

Integrating in polar is also slightly different. In the Cartesian coordinate plane, we integrate by taking the sum of an infinite number of rectangles with an infinitely small width and some length $f(x)$. In polar, we take the sum of an infinite number of wedges, or sectors. These sectors have an area $\pi\cdot r^2\cdot\frac{d\theta}{2\cdot\pi}$, which means that our function, when integrated, looks like this:

$$\int \frac{(f(\theta))^2}{2}\cdot d\theta$$

Note that sin functions and some other functions in polar cover their area twice; that is to say, they complete a full rotation from 0 to π, not 0 to 2π. This should be taken into account when integrating, which is another reason its important to draw your function before you graph it.

The last topic we covered was differential equations. Differential equations are written as a function of y. This means differential equations speak about the rate of change being equal to some constant multiplied by itself. The solution to a differential equation can be found by two methods: seperation of variables and series.

First, let's look at a differential equation. The differential equation for population capacity is $y' = k \cdot y \cdot (C - y)$ where C is the carrying capacity, or the maximum number of things capable of living in a given area. You should be able to speak about a differential equation in English, without reffering to the variables; for example, this differential equation says that the rate of growth in a population is equal to a constant times the number of people times the quantity of the carrying capacity minus the number of people.

Slope fields are a graphical representation of differential equations. They represent an infinite number of possibilities as to what the solution could be, but provide a general trend that gives us clues as to how the solution may work.

Now let's look at the acutal solution to this differential equation, $y' = k \cdot y \cdot (C - y)$. By seperation of variables, we follow this procedure:

$$y' = k \cdot y \cdot (C - y)$$

$$\frac{dy}{dx} = k \cdot y \cdot (C - y)$$

$$\frac{dy}{k \cdot y \cdot (C - y)} = dx$$

$$\int \frac{dy}{k \cdot y \cdot (C - y)} = \int dx$$

$$\frac{ln|y|}{C} - \frac{ln|C - y|}{C} = k \cdot x + J$$

Note here that the constants of integration were combined onthe right side and written as J, not C, as C already reprsents the carrying capacity.

$$ln|y| - ln|C - y| = C \cdot k \cdot x + J \cdot C$$
$$ln|\frac{y}{C - y}| = C \cdot k \cdot x + J \cdot C$$
$$|\frac{y}{C - y}| = e^{C \cdot k \cdot x} + e^{J \cdot C}$$
$$|\frac{y}{C - y}| = A \cdot e^{C \cdot k \cdot x}$$
$$\frac{y}{C - y} = A \cdot e^{C \cdot k \cdot x}$$

Note that before this step, A was positive. We can remove the absolute value signs by saying that now A can be positive or negative.

$$y + y \cdot A \cdot e^{C \cdot k \cdot x} = A \cdot C \cdot e^{C \cdot k \cdot x}$$

Solving for y gets us the solution to the differential equation:

$$y = \frac{A \cdot C \cdot e^{C \cdot k \cdot x}}{1 + A \cdot e^{C \cdot k \cdot x}}$$

Another important differential equation is Newton's Law of Cooling: $y' = k \cdot (y - T_a)$ where T_a is the ambient or room temperature. I won't show the process for solving for the solution, but the solution that you should get after using seperation of variables is $y = A \cdot e^{k \cdot x} + T_a$.

With these differential equations, its not always necessary to solve for the general solution; maybe you just need to get an approximation of the next few terms. In this case, we can Euler's approximation. To do this, figure out what your change in x is (usually 1). Take your starting y value and substitute it into the differential equation, along with the x value.

Note that this assumes the only variables you have in your differential equation are x and y. If you have constants, you may need to solve for them with the given information.

Once you get a value from your differential equation, add that to your "old y", the y value you plugged into the differential equation. This will result in your new y. Your x value will increase by whatever step you're taking, and you'll be able to repeat the process. This method is an approximation; you can find the error by finding the solution, and solving for whatever y value you solved for with the approximation, and subtracting the two.

The other method of solving a differential equation shouldn't be necessary, but I'll overview it briefly. Using series to solve a differential equation is slightly complex. First, we assume that y can be represented by some general power series, $\sum_{k=0}^{\infty} a_k \cdot x^k$. Let's say we're trying to solve for the differential equation $y'' = C \cdot y$. We can differentiate y, and the power series, to find the power series for y''. We know that these two are equal when y is multiplied by the constant C, or when the coefficients of x in the power series are equal. So we can set up a table equating the coefficients of the various powers x between the two power series. After doing this for several powers of x, try to find a pattern that equates the coefficients. This should allow you to solve for a_k in terms of C, after which you can substitute back into the differential equation. Note that while doing thi,s the power series you come up may be adaptations of power series you're familiar with (like sin or cos). This means you can substitute these functions in to get a solution that involves no series.

I doubt this will be on the test. But it's a very logical method - get a power series representation of the differential equation, find out how to get the coefficients equal to one another, and substitute the power series out by replacing them with functions.

This concludes the review of all of the concepts that we'll probably be tested on during the final. If you have any questions, email me at kyarlagadda@imsa.edu. If you find any errors in this guide, or think there's something I should add, let me know as well. Dr. Fogel and Dr. Prince probably covered material in different ways, so if the way one of them explained things to you makes more sense then the way I present it here, don't listen to me. Go with whatever works for you. Good luck on the final.

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